Undecidability of Macroscopically Distinguishable States in Quantum Field Theory

ARTHUR KOMAR*
Syracuse University, Syracuse, New York, and Yeshiva University, New York, New York

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A heuristic discussion is presented regarding quantum field theory as a synthesis of the complementary theories of classical mechanics and quantum mechanics. If the states of quantum field theory are partitioned in equivalence classes accordingly as their occupation numbers differ in a finite or an infinite number of places, it is suggested that we define states to be macroscopically distinguishable if they belong to different equivalence classes. It is then proven that there is, in general, no effective procedure for determining whether or not two arbitrarily given states of a quantum system having an infinite number of degrees of freedom are macroscopically distinguishable.

I. INTRODUCTION

THERE is a viewpoint prevalent among a wide group of physicists that quantum theory provides a correct (or more nearly correct) theory of nature, and that classical mechanics is merely an approximation obtained by going to the limit of vanishing Plank's constant, in much the same way that nonrelativistic mechanics is obtainable as a limit of special relativistic mechanics as we take the velocity of light to be arbitrarily large. There is no question that the dynamical laws of classical and quantum theory are strikingly parallel, and, in fact, can be understood to be identical when expressed in terms of the group theoretic commutators of the generators of infinitesimal translations and rotations. However, the kinematics of the two theories are vastly dissimilar, and it is simply not true that in the limit of vanishing Plank's constant a vector in a Hilbert space becomes a point in a classical phase space. The limiting theory, which one obtains from the theory of a single quantum system by allowing Plank's constant to vanish, corresponds more nearly to a theory of an *ensemble* of classical systems. It is in this more limited sense which we may understand classical mechanics as a limit of quantum mechanics. However, the relationship between classical and quantum theory is even more intricate. For not only is classical theory a limit (in the above sense) of quantum theory, it is essential for the physical interpretation of the mathematical symbols employed in the quantum theory.[1] This dual role which classical mechanics plays relative to quantum mechanics indicates that the relationship between these two theories is of a rather different character than is the relationship between nonrelativistic and special relativistic mechanics.

The viewpoint which is suggested is to regard classical mechanics and quantum mechanics as complementary theories, each with its own domain of validity and each necessary for a full understanding and interpretation of the other. The question immediately arises whether such a program can be carried out, and, if so, what precisely are the domains of validity of the two theories. In some intuitive sense, what is required for the specification of the domains of validity is a criterion for determining or specifying a distance between states with the property that states which are "near" to each other can be expected to linearly superpose and interfere, while for states which are "far apart," the relative phases are meaningless, and, therefore, there would be no possibility of observing the characteristic quantum phenomenon of interference. In the latter situation, one may regard such states as macroscopically distinguishable and might expect that their relative behavior to be governed by the laws of classical mechanics.

For systems having a large number of degrees of freedom, H. Wakita[2] has suggested that phase relations between states which differ from each other in many degrees of freedom should not be meaningful in the sense that there should be no observables having matrix elements between such states. Such a suggestion would be in the nature of an approximate superselection rule for the permissible Hermitian operators. While highly intuitive, it is not clear whether such a prescription for specifying distances between states can be carried out in a representation-invariant manner.

For systems having an infinite number of degrees of freedom a much more natural and satisfactory situation arises.[2] The vector space for quantum field theory is no longer separable. If we form equivalence classes of states by placing two states in the same equivalence class if they differ in at most a finite number of degrees of freedom (that is, if one state can be obtained from the other by the application of at most a finite number of creation and annihilation operators), we then know that there are no unitary mappings of equivalence classes onto each other,[3] and that the relative phases between states belong to different equivalence classes are meaningless. It would, therefore, appear natural to regard states belonging to different equivalence classes as macroscopically distinguishable. Such macroscopi-

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[1] N. Bohr, *Atomic Theory and the Description of Nature* (Cambridge University Press, New York, 1961).

[2] H. Wakita, Progr. Theoret. Phys. (Kyoto) **23**, 32 (1960).

[3] A. S. Wightman and S. S. Schweber, Phys. Rev. **98**, 812 (1955).

cally distinguishable states would then have the intuitively desirable property of differing from each other in infinitely many degrees of freedom. Our view of quantum field theory as it now emerges would appear as a synthesis of the complementary theories of classical mechanics and quantum mechanics in much the same way that quantum mechanics unified the complementary theories of classical waves and classical particles. More specifically, it would appear that the domain of quantum mechanics is within each equivalence class, whereas classical mechanics would be more appropriate for the mechanics of the equivalence classes themselves.

That quantum field theory may contain important elements of classical mechanics is also suggested by another heuristic approach. It can be shown[4] that classical mechanics can be written in a form which, purely formally, appears to be quantum mechanics. In order to obtain the specifically quantum properties of discrete spectra it is necessary to impose the additional (and not very natural) boundary conditions that the state vector be square integrable. Unless this is done, the theory remains essentially an odd version of classcal mechanics, or, as Schiller prefers to call it, quasiclassical mechanics. It is, therefore, not unreasonable to expect that by focusing our attention on the equivalence classes of quantum field theory, the state vectors of almost all the classes being in fact not normalizeable, we can reveal a classical or perhaps quasiclassical aspect of the theory.

II. UNDECIDABILITY

The foregoing discussion and outline of a program will not be pursued further in the present paper. It was intended partly to motivate the introduction of the suggestive terminology of "macroscopically distinguishable states," and partly to indicate a possible broad epistomological significance to the precise result which we shall develop in the present section. For those readers who take issue with the discussion in the introduction, if they would kindly read "states which belong to different equivalence classes" whenever I employ the expression "macroscopically distinguishable states," they will find the remainder of the paper quite independent of the introduction.

Suppose we wish to specify a particular state of a quantized field. Mathematically, this could be accomplished by giving the occupation numbers for each degree of freedom in a particular representation. That is, we would, in general, have to give an infinite sequence of integers since we are considering systems having infinitely many degrees of freedom. (For fermions the sequences would only consist of zeros and ones, whereas for bosons all integers may occur, but this distinction is immaterial for our considerations.) Thus, a state may be "named" by an infinite sequence of integers, or equivalently by a function from the integers to the integers. Physically, a state can be specified by describing its preparation. That is, in ordinary language one must specify in detail the construction and arrangement of the apparatus which is to prepare the state. If the system under consideration truly has infinitely many degrees of freedom, the description of the apparatus must be so detailed that it effectively assures us the possibility of determining a precise infinite sequence of integers. Of course, we cannot "name" an infinite sequence by writing down all its terms. All that we really require is a recursive procedure for determining for example, the $n+1$st term in the sequence possibly given the first n terms. Thus, a state is effectively "named" by giving a recursive function from the integers to the integers. The precise statement of the physical preparation of the state should yield in effect a recursive procedure for determining the same function.

Consider, now, the following problem: We are given two states of a quantized field, named either by the precise specification of their physical preparation, or by giving the recursive procedures required to determine their sequence of occupation numbers in a given representation, and we wish to know whether the two states are macroscopically distinguishable. That is, we wish to know if it is meaningful to expect the two states to interfere in a superposition, or if the relative phases of the two states are meaningless. Should there exist a procedure for enabling one to decide the macroscopic distinguishability of two arbitrary states, such a procedure would yield a method for determining of two infinite sequences of integers whether they differ in at most a finite number of places. Equivalently, such a procedure would enable us to decide of a function $f(a)$ obtained by forming the difference between two arbitrary recursive functions from the integers to the integers whether there exists a finite integer n such that $f(a)=0$ for all $a>n$. *This decision problem is known to be recursively unsolvable.*[5]

III. CONCLUSIONS

Given two arbitrary quantum states of a physical system having an infinite number of degrees of freedom, we know either that they are macroscopically distinguishable, or that they are not; for, either they lie in the same equivalence class of states or they do not. What we have shown is that there is in general no effective procedure for deciding whether they do or not. That is, there exists no effect procedure for determining whether two arbitrarily given physical states can be superposed to show interference effects characteristic of quantum systems. (Of course, in individual special cases the answer may be evident.) It is not at all clear what

[4] R. Schiller, Phys. Rev. **125**, 1100 (1962).

[5] N. Shapiro, Trans. Am. Math. Soc. **82**, 281 (1956); cf. M. Davis, *Computability and Unsolvability* (McGraw-Hill Book Company, Inc., New York, 1958), p. 172.

effect this result may have on the feasability of effectuating the program outlined in the introduction.

Although it has long been know from the Gödel incompleteness theorem[6] that any logical system sufficiently complicated to contain the integers is either incomplete or inconsistent, it is rather curious or surprising that the issue of the macroscopic distinguishability of quantum states should be among the undecideable questions. It should be pointed out that this result depends critically on the system having an infinite number of degrees of freedom. This is evident for two separate reasons: (1) The construction of equivalence classes, essential for the criterion of macroscopic distinguishability, made essential use of the existence of an infinite number of degrees of freedom; and (2) the proof in Ref. 5 of the unsolvability of the decision problem requires that the function is defined on an infinite domain, for it is evident that for functions defined on a finite domain one can in principle "name" them by exhibiting all their values.

[6] K. Gödel, Monatsh. Math. Phys. **38**, 173 (1931).

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